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## CHAPTER 17

# DEVELOPMENTS IN LOGIC: CARNAP, GÖDEL, AND TARSKI

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ANALYTIC philosophy and modern logic are intimately connected, both historically and systematically. Thinkers such as Frege, Russell, and Wittgenstein were major contributors to the early development of both; and the fruitful use of modern logic in addressing philosophical problems was, and still is, definitive for large parts of the analytic tradition. More specifically, Frege's analysis of the concept of number, Russell's theory of descriptions, and Wittgenstein's notion of tautology have long been seen as paradigmatic pieces of philosophy in this tradition. This close connection remained beyond what is now often called 'early analytic philosophy', i.e., the tradition's first phase. In the present chapter I will consider three thinkers who played equally important and formative roles in analytic philosophy's second phase, the period from the 1920s to the 1950s: Rudolf Carnap, Kurt Gödel, and Alfred Tarski.

Undoubtedly, Gödel and Tarski were two of the greatest logicians of the twentieth century, indeed of all time. Their influence on mathematical logic can hardly be exaggerated. However, they also exerted a significant influence on philosophy, especially analytic philosophy, as we will see. Carnap, in turn, is widely recognized as one of the most important and influential analytic philosophers of the twentieth century. Yet he also played an under-appreciated role in the history of modern logic. For a number of reasons it makes sense to treat these three figures together: they were all born and educated in Central Europe (in Germany, Austria, and Poland, respectively); they knew each other personally, from early on in their careers; they interacted frequently and influenced each other's views directly; and all three emigrated to the United States, within a few years of each other, so as to exert their main influence there.

In briefly surveying the contributions of three such seminal figures it is impossible to be comprehensive; thus my discussion will have to be selective. My main objective will

be to establish the major role their contributions to logic played in the development of analytic philosophy (as opposed to, say, the development of mathematical logic). Thus, I will focus on their philosophically most influential results. Various technical details will be suppressed and a wide range of their other achievements only mentioned in passing. My discussion will revolve around the following related topics: the transformation of modern logic, especially the rise of meta-logic; logicism and its relation to formal axiomatics; the notions of truth, logical truth, and logical consequence; formal semantics, metaphysics, and epistemology; and philosophical methodology. A recurring theme will be Carnap's, Gödel's, and Tarski's continued interactions, which will reveal many shared interests, but also some striking differences in their philosophical convictions.

## 17.1 FIRST ENCOUNTERS AND INITIAL INTERACTIONS

The first time all three of our protagonists met was in Vienna in February 1930. The occasion was Tarski's first visit to the city. On the invitation of the mathematician Karl Menger, Tarski was to give three talks at the University of Vienna. Their topics were: (i) set theory, (ii) methodology of the deductive sciences, (iii) the sentential calculus. A main motivation for Menger's invitation had been to establish closer ties between the Vienna Circle, of which he was a member, and the Lvov-Warsaw school of logic, for which Tarski served as a kind of emissary. These two groups (as well as the Berlin group around Hans Reichenbach) shared a preference for 'scientific philosophy'. This meant: the rejection of grand, speculative system building and its replacement by more specific, detailed analyses of concepts; the focus on philosophical questions arising out of the exact sciences; and the application and further development of modern logic. Tarski's Vienna talks exemplified all of these features, but especially the third.

Alfred Tarski (1901–83) had received his Ph.D. in mathematics from the University of Warsaw in 1924, with a dissertation on logic under Stanisław Leśniewski. Among his other teachers were: Tadeusz Kotarbiński in philosophy, Jan Łukasiewicz in logic, and Wacław Sierpiński in set theory. Tarski was, in fact, their star student, which is why he had been selected to represent 'Polish logic' in Vienna. By the late 1920s he had already arrived at important results in several subfields of mathematical logic, including: set theory (on uses of the Axiom of Choice, especially the well-known Banach-Tarski Paradox), general axiomatics (new axiomatizations for geometry), on the decision problem (decision procedures for elementary geometry and algebra), and on the topic of definability (concerning definable sets of reals). Tarski had also started to investigate the 'methodology of the deductive sciences' more generally; and indeed, this served as

the theme of his second talk in Vienna. This is the talk that attracted Carnap's attention the most, and it was soon after it that the two had their first substantive discussion.<sup>1</sup>

Ten years older than Tarski, Rudolf Carnap (1891–1970) had received a Ph.D. in philosophy from the University of Jena in 1921, with a dissertation on geometry (published, as *Der Raum*, in 1922). Besides his dissertation adviser Bruno Bauch and the writings of other neo-Kantian philosophers, there had been further formative influences on Carnap: in 1910–14, while at Jena, he took classes in logic and the foundations of mathematics from Gottlob Frege; in the early 1920s, after finishing his dissertation, he read Bertrand Russell's logical works in detail, including Whitehead and Russell's *Principia Mathematica*; and in 1921–2, he took classes with Edmund Husserl at the University of Freiburg. Carnap arrived at the University of Vienna in 1926, hired as a senior lecturer (*Privatdozent*) and with a draft of *Der Logische Aufbau der Welt* in hand (his *Habilitation*, published in 1928), the text that established him as a major thinker among scientific philosophers. He started to take part in activities of the Vienna Circle right away, but continued other projects as well. Two results were: *Pseudoprobleme in der Philosophie* (also published in 1928) and *Abriss der Logistik* (1929).<sup>2</sup>

Kurt Gödel (1906–78), finally, was the youngest of our three thinkers, five years younger than Tarski and fifteen years younger than Carnap. (In 1930, they were 24, 29, and 39 years old.) At the time of Tarski's arrival he had just finished his Ph.D. in mathematics at the University of Vienna, under Hans Hahn. In addition to working with Hahn, Menger, and the number theorist Philipp Furtwängler, Gödel had taken classes in the philosophy of mathematics and logic from Moritz Schlick and, especially, from Carnap, which awakened his interest in foundational studies. He had also been drawn into the activities of the Vienna Circle more generally, although he remained quietly independent in his philosophical convictions. Gödel's dissertation, accepted in 1929, already contained a major result in logic: a proof of the completeness of first-order logic. It answered a question posed by David Hilbert, in an influential lecture in Bologna (1928) and in Hilbert and Ackermann's *Grundzüge der theoretischen Logik* (1928). It was also directly connected with Carnap's work in logic (as we will see more below). During Gödel's and Tarski's first meeting in Vienna, this is the result they discussed.<sup>3</sup>

After their initial meeting, in February 1930, there would be many further interactions between the three thinkers, later that year and subsequently. For example, in October 1930 Carnap and Gödel attended the well-known Königsberg conference on the foundations of mathematics together. At that occasion, Carnap gave a talk on logicism; Gödel presented his completeness result, and to everyone's surprise, he also announced his incompleteness theorem for *Principia Mathematica* and related systems. Actually, Gödel had already told Carnap about the latter in August 1930, during

<sup>1</sup> For more on Tarski's life and his (early as well as later) works, cf. Feferman and Feferman (2004). For details concerning his publications, see the references.

<sup>2</sup> For more on Carnap's background and early works, cf. Reck (2004), the first half of Creath and Friedman (2007), and Carus (2007). Concerning his publications, see again the references.

<sup>3</sup> For Gödel's background, life, and (early as well as later) works, cf. Dawson (1997). For his relevant publications, see again the references.

conversations in Vienna. He informed Tarski, who was not present in Königsberg, of its details in a letter in January 1931. As another example, Tarski invited Carnap to come to Warsaw in November 1930, as an emissary of the Vienna Circle, to present three talks at the university there in turn. But it was the first meeting of the three of them that set the stage for many of their later interactions. To understand better how and why, let us briefly review the development of logic and related issues up to 1930.

## 17.2 LOGIC, LOGICISM, AND AXIOMATICS UP TO 1930<sup>4</sup>

Modern logic is often taken to start with Frege's *Begriffsschrift* (1879). In it, both propositional and quantificational logic are presented systematically for the first time, in the form of a simple theory of types (a form of higher-order logic). They are also used to analyse a core part of the foundations of arithmetic, the principle of mathematical induction, thus inaugurating Frege's logicism—his project of reducing arithmetic to logic alone. He motivated this project further in *Die Grundgesetze der Arithmetik* (1884), and he expanded on both his technical machinery and its application in *Grundgesetze der Arithmetik*, Vols. I–II (1893/1903). In the latter, he added a theory of classes ('extensions of concepts') to his logic that, as is well known, falls prey to Russell's antinomy. Partly for that reason, Frege's work was largely ignored for a while, although not entirely, as Carnap learned about it in his Jena classes. A few other thinkers were directly influenced by his contributions as well, most crucially Russell and Wittgenstein.

Nevertheless, it was Bertrand Russell's subsequent writings on logic, and especially A. N. Whitehead and B. Russell's *Principia Mathematica*, Vols. 1–3 (1910–13), that had a much more widespread influence. Indeed, virtually everyone concerned about modern logic in the first half of the twentieth century studied *Principia*, including Carnap, Gödel, and Tarski. As in Frege's case, Russell's logical system contained a theory of classes (at least indirectly, in the form of a 'no-classes theory of classes'). The general framework was a ramified theory of types (a more complex version of higher-order logic), introduced to avoid a whole range of antinomies discovered by then (not just 'set-theoretic', but also 'semantic' antinomies such as Richard's). In addition, Russell's logicist aspirations were more far-reaching than Frege's: he saw logic as the foundation for all of mathematics, not just arithmetic. Suitably supplemented, it was even to form a framework for all scientific knowledge, as sketched in Russell's *Our Knowledge of the External World* (1914).

Another crucial development for our purposes, initially separate from Frege–Russell logic, is the emergence of modern axiomatics. It grew out of the investigation of various non-Euclidean geometries in the nineteenth century, but led to a reconsideration of

<sup>4</sup> In the next three sections, I draw heavily on Awodey and Carus (2001), Awodey and Reck (2002a), Reck (2004), and Reck (2007). For mathematical details, cf. also Mancosu, Zach, and Badesa (2009).

Euclidean geometry as well, culminating in Hilbert's *Grundlagen der Geometrie* (1899). There were also related novel treatments of arithmetic by Dedekind and Peano, of analysis by Dedekind and Hilbert, and of set theory by Zermelo (the latter along Cantorian and Dedekindian lines, in contrast to Frege's and Russell's logicist theories of classes). This intense focus on axioms, together with Hilbert's 'formalist' rethinking of them, led to questions about the independence, consistency, and completeness of the main axiom systems. These were investigated in Hilbert's Göttingen, earlier also by the 'American Postulate Theorists', E. Huntington, O. Veblen, etc.<sup>5</sup> Another issue that became prominent during this period was the mechanical decidability of the corresponding parts of mathematics (the *Entscheidungsproblem*).

In pursuing such issues, it gradually became clear that one has to be mindful of the logical system in which one works. For one thing, there is a difference between the completeness of a mathematical axiom system (the issue of whether it 'decides' all relevant sentences), and the completeness of the logic in the background (whether it allows for formal, syntactic proofs of all semantic consequences). For another, while Russellian ramified type theory was used initially as the proper logical framework, this is not the only option; e.g., one can use Fregean simple type theory instead (stripped of its inconsistent theory of classes, as Frege had done in the lectures Carnap attended). Indeed, the simplicity of Frege's version of higher-order logic seemed preferable for various purposes, as Frank Ramsey and others began to argue. Along such lines, Carnap's early logic textbook, *Abriss der Logistik* (1929), abandoned Russellian ramifications; similarly for Hilbert and Ackerman's *Grundzüge der theoretischen Logik* (1928), at least in its later editions. Moreover, within simple type theory certain self-contained subsystems can be isolated and studied profitably, especially propositional logic and first-order logic. With this proliferation of logical systems and subsystems, the question arose: Which of them, if any, should be seen as 'the correct' logic?

Already in the 1910s, special attention to propositional logic led to proofs, by Paul Bernays (1918) and Emil Post (1921), of its completeness.<sup>6</sup> The natural next step concerned the completeness of first-order logic (the 'lower functional calculus', as it was called at the time)—as established in Gödel's dissertation, 'Über die Vollständigkeit des Logikkalküls' (1929). After that, the completeness of the simple theory of types as a whole remained as a question.<sup>7</sup> Parallel to these developments in logic, various systems of mathematical axioms were investigated in more detail, either as formulated in first-order or higher-order logic. The case of first-order axiomatic set theory, based on suggestions by Skolem, Weyl, and Fraenkel (but resisted by, e.g., Zermelo), attracted much attention, from the 1920s on. Axiom systems for the natural numbers, the real numbers, and various parts of geometry were studied in novel ways as well.

<sup>5</sup> For more on the rise of 'formal axiomatics' and 'postulate theory', see Awodey and Reck (2002a).

<sup>6</sup> In the case of propositional logic, completeness amounts, more explicitly and precisely, to the existence of an adequate (strong enough) deduction system relative to truth-value semantics.

<sup>7</sup> For first- and higher-order logic, including simple type theory, completeness means here the existence of an adequate (strong enough) deduction system relative to standard set-theoretic semantics.

In Hilbert's school the consistency of such axiomatic theories was explored with great vigour. The topic of decidability was seen as closely related, as some decision procedures, applicable in restricted contexts, were discovered. Concerning set theory, there were also questions about the legitimacy of specific axioms, particularly the Axiom of Choice (after its explicit formulation by Zermelo in 1904). All of this fed into Hilbert's 'meta-mathematical' and 'proof-theoretic' programs, which took shape in the late 1910s and early 1920s.<sup>8</sup> The strong focus on consistency was partly a response to the antinomies already mentioned, which were widely seen as leading to a 'foundational crisis'. Additional pressure came from intuitionistic or constructivist mathematicians, especially Brouwer and (for a while) Weyl, who rejected both a formal axiomatic approach and highly non-constructive principles such as the Axiom of Choice. The opposition between the classic schools of logicism, formalism, and intuitionism resulted. In connection with formalism, Hilbert was led to 'finitist' restrictions of the means by which consistency proofs were to be given, so as to convince even intuitionists of their cogency.<sup>9</sup>

Concerning logicism, two further issues arose in the 1920s. The first involves a controversial aspect of *Principia Mathematica*: its reliance on the axioms of infinity and reducibility, introduced somewhat *ad hoc* so as to be able to derive all of classical mathematics. Neither of them could easily be accepted as logical, which led to the question of how to justify their use. Second, there was Wittgenstein's new notion of tautology, introduced in his *Tractatus Logico-Philosophicus* (1921). It was presented as filling a gap in Frege's and Russell's works: to provide a precise, general characterization of logical laws. In Wittgenstein's and others' eyes, this notion had the additional advantage of providing a deflationary account of logical truth (as a tautology is true in virtue of its form alone, no matter what other facts obtain). And that led to the question of how far such an account could be extended, since its only clear application was to propositional logic (pace Wittgenstein's further claims). Both of these issues were much discussed at the time, including in the Vienna Circle of the 1920s.

## 17.3 CARNAP'S AND TARSKI'S EARLY FORAYS INTO META-LOGIC

The developments just described indicate that the 1920s were an extremely fertile period in the history of logic. (Together with the 1930s, they formed modern logic's 'Golden Age'.) This is further confirmed if we add Carnap's and Tarski's contributions to the mix. It should be evident that much of Tarski's early work, as mentioned above (on the Axiom of Choice, new axiomatizations for geometry, decision procedures for

<sup>8</sup> For more on the early development of Hilbert's program, or programs, cf. Sieg (1999); see also Zach (1999).

<sup>9</sup> For a relatively recent comparison of the three main schools, cf. George and Velleman (2002).

elementary geometry and algebra, etc.), fits squarely into these developments. The same holds for some of Carnap's works from the 1920s, especially *Abriss der Logistik*, but also *Der Logische Aufbau der Welt*, a book motivated by Russell's suggestion (as well as related neo-Kantian and, to some degree, Husserlian ideas) to logically reconstruct scientific knowledge in general.<sup>10</sup> But the main focus of Carnap's and Tarski's initial conversation in Vienna, in 1930, was somewhat different. It concerned what Tarski liked to call 'the methodology of the deductive sciences', what was called 'meta-mathematics' in the Hilbert school, and what often goes under the name of 'meta-logic' today.

Carnap was led to meta-logical considerations in at least four ways (against the background of the general developments already mentioned). First, from early on in his career he was not only exposed to Frege–Russell logic, but also to Hilbertian axiomatics. Especially relevant among Hilbert's writings, in addition to *Die Grundlagen der Geometrie*, was the article 'Axiomatisches Denken' (1918), which advocated a wide-ranging application of the axiomatic method. Second, while in Frege's logic classes at Jena, as well as in Russell's writings, Carnap had been confronted with their critical, even dismissive attitude towards axiomatics, he did not take over that attitude. Instead, his reaction was to strive for a reconciliation and synthesis, i.e., he wanted to combine Frege–Russell logic with a Hilbertian axiomatic approach. Third, both logic and the axiomatic method played a prominent role in the broader discussions of scientific knowledge in the Vienna Circle, as illustrated by Schlick's remarks on 'implicit definitions' in *Allgemeine Erkenntnislehre* (1918, second edition 1925) and Carnap's response in 'Eigentliche und Uneigentliche Begriffe' (1927). Fourth, in the mid-1920s Carnap encountered Abraham Fraenkel's work on axiomatics, which suggested a potentially fruitful way of approaching crucial logical and meta-logical issues in all of these connections.<sup>11</sup>

The text by Fraenkel that influenced Carnap the most was his *Einführung in die Mengenlehre*, especially its second edition (published in 1923) to which a novel section on 'general axiomatics' had been added. This section contained a probing discussion of the notion, or of several related notions, of completeness for axiomatic systems. Carnap quickly started a correspondence with Fraenkel about this topic. One result of it, as evident from the third edition of Fraenkel's book (1928), was the sharper differentiation and characterization of three notions of completeness that had often been confused or simply identified so far, namely: 'syntactic completeness', 'semantic completeness' (being 'non-forkable', as Carnap called it), and 'categoricity' (being 'monomorphic').<sup>12</sup>

As Fraenkel also stressed, the relationships between these three notions were in need of further exploration. In order to make progress with that task one had to go beyond the informal setup of Fraenkel's book—one had to make explicit the logical framework in which one intended to work. Realizing that, Carnap had a specific suggestion: use the simple theory of types, as spelled out in his *Abriss der Logistik*.

At this point Carnap had a new research project at hand and started to compose another book, with the working title *Untersuchungen zur Allgemeinen Axiomatik*. Pretty quickly he produced a partial manuscript, which he then, from 1928 on, circulated among friends—including Gödel. Carnap actually thought he had arrived at substantive results already, i.e., proved several core theorems in general axiomatics. Specifically, he believed he had proved that, within simple type theory, all three notions of completeness just distinguished were equivalent. He also believed that, within the same context, any consistent axiom system was satisfiable (has a model)—a version of the completeness of simple type theory. He was mistaken on both counts, and there were other, more basic problems with his approach. It was exactly those basic problems he was confronted with in his first meeting with Tarski, in February 1930. And Gödel's incompleteness theorem, of which he heard in August 1930, confirmed that something fundamental was amiss. Right after realizing that he gave up the *Allgemeine Axiomatik* project, but only after having interacted significantly with both Gödel and Tarski on its basis.<sup>13</sup>

In the next section I will explain how exactly Gödel's surprising theorem undercut Carnap's project. To close off this section, let me say a bit more about the problems of which Tarski made him aware. Basically, what Carnap had done in his *Allgemeine Axiomatik* manuscript was to work within one logical system and define all the relevant notions internal to it. (He was still a Fregean or Russellian 'universalist' in that respect.) But what the situation really called for was to work at two distinct logical levels and with two different languages: the level of the given axioms and of the deductive system in the background, as formulated in an 'object language'; and the level at which results about them were established, in a 'meta-language'. That is to say, Carnap was learning the hard way, through his failures, that the issues he was interested in could only be captured adequately, and investigated properly, by proceeding 'meta-logically'. Tarski had already achieved considerable clarity on such matters in his seminars at the University of Warsaw in the 1920s, but this was little known beyond Poland at the time. And indeed, these were exactly the kinds of achievements to be disseminated more widely in Tarski's second Vienna talk, on 'the methodology of the deductive sciences'.

While the details of Tarski's talk seem not to have been preserved, one can get a sense of its content from articles he published around the same time, such as: 'Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften' (1930a) and 'Über einige fundamentale Begriffe der Metamathematik' (1930b). His general topic in them is

<sup>13</sup> Carnap's corresponding manuscript was not published until seventy years later, as Carnap (2000). It should be added that, as his project led to some partial results, it was not a complete failure; cf. Awodey and Reck (2002a) and Reck (2007). Tarski addressed closely related issues in Tarski and Lindtenbaum (1934–5).

<sup>10</sup> For more on the big role Carnap's *Aufbau* played in the history of analytic philosophy, cf. Richardson (1998), Friedman (1999), the relevant pieces in Creath and Friedman (2007), and Carus (2007 chs. 5–6).

<sup>11</sup> Concerning the first, second, and fourth aspects, compare Reck (2004) and (2007). Concerning the third—which deserves more attention than it has received so far and than I can give it here—cf. Goldfarb (1996), several of the articles in Friedman (1999), Awodey and Carus (2001), and Carus (2007, ch. 7).

<sup>12</sup> Briefly, an axiom system is syntactically complete if, for each sentence in the given language, either it or its negation is deducible from the axioms (i.e., follows syntactically from them); it is semantically complete if, for each such sentence, either it or its negation is true in all models of the axioms (i.e., follows semantically); and it is categorical if all models of the axioms are isomorphic. Compare Awodey and Carus (2001), Awodey and Reck (2002a), and Reck (2007) for further discussion and background.

the 'deductive method', and the main goal is to clarify how best to organize 'deductive theories', including mathematical theories, so as to study their properties more precisely. This involves distilling out basic concepts and axioms, but also making explicit—along meta-logical lines—the notions of definition, sentence, consequence, theory, etc. On that basis, Tarski was able to establish results about definability, axiomatizability, independence, consistency, and completeness. The connections to Hilbertian meta-mathematics, on the one hand, and to the general axiomatics pursued by Fraenkel and Carnap, on the other, are clear. It should be added that, while Tarski was ahead of Carnap in various respects, his approach would require later modifications as well. Both were working at the cutting edge of logic, where things were still in flux.<sup>14</sup>

## 17.4 GÖDEL'S INCOMPLETENESS THEOREMS

I already mentioned that Gödel's first major result in logic, his completeness theorem for first-order logic, was answering a question prominently raised by Hilbert. His second result, the incompleteness theorem (or theorems), can also be seen as a response to Hilbert, specifically to his goal of proving the consistency of classical mathematical theories by restricted means. Indeed, this is how the result is typically discussed in the literature. We are now in a position to recognize, however, that the theorem can equally well be seen as a response to Carnap's work in logic from the 1920s, of which Gödel knew first-hand. (Not only did he get a copy of *Allgemeine Axiomatik* in 1928, he read *Abriss der Logistik*, a text circulated in Vienna from 1927 on, and he attended lectures on logic and the foundations of mathematics by Carnap during the period.) It was no accident, then, that Carnap was one of the first people to be told about this discovery by Gödel, even before he announced it publicly at the Königsberg conference.

The title of Gödel's famous paper on the subject, 'Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I' (1931), points towards Russellian ramified type theory as the relevant logical framework. But in fact, Gödel worked with the simple theory of types in this paper—in line with Carnap. What he showed was that any axiomatic theory formulated in simple type theory that contains (a moderate amount of) arithmetic does not allow one to decide, by formal deductions from its axioms, all the sentences in its language; there is always a sentence such that neither it nor its negation is deducible (assuming consistency). In other words, any such theory is syntactically incomplete. And this result applies quite widely: to arithmetic; to analysis; to simple and ramified type theory with a theory of classes; to Zermelo-Fraenkel set theory; etc. Moreover, if one tries to amend things by adding new axioms to the theory (while meeting a minimal condition

of adequacy, namely their recursive enumerability, as before), the same situation recurs: there will again be a sentence that is neither deducible nor refutable within the enlarged system. This is, basically, the content of Gödel's 'first incompleteness theorem'. According to his 'second incompleteness theorem', which follows from the first, the consistency of such theories can also not be proven by elementary means (on pain of inconsistency).<sup>15</sup>

Another way in which Gödel's first incompleteness theorem is often put is that no formal system (that satisfies certain minimal conditions, including consistency) can capture all of arithmetic—arithmetic is 'inexhaustible'. And consequently, mathematics as a whole is inexhaustible in the same sense. Gödel's first incompleteness theorem also implies that higher order logic, either in the form of simple or ramified type theory, is not complete in the sense in which propositional logic and first-order logic are complete.<sup>16</sup> For anyone working with an axiomatic approach along Hilbertian lines these are striking results, to say the least. The same holds for logicians working along Fregean or Russellian lines. Gödel's second incompleteness theorem adds that Hilbert's goal of establishing the consistency of central mathematical theories by restricted means is futile. In particular, no consistency proof for arithmetic can be given that doesn't employ resources stronger than arithmetic itself; likewise for set theory, etc.

Gödel's results undermine Carnap's *Allgemeine Axiomatik* project in two ways. First, Carnap thought he had proved the completeness of simple type theory, which, as just noted, cannot be the case (and his supposed proof was, in fact, flawed). Second, he believed he had established that, within simple type theory, the syntactic completeness of any axiomatic theory is equivalent to the theory's semantic completeness. Now, it is a standard theorem, and one that was well known at the time, that the higher-order axioms for arithmetic—the Dedekind-Peano axioms as formulated within simple type theory, say—are semantically complete, since categorical. Yet according to Gödel's first incompleteness theorem these axioms are not syntactically complete, as there are formally undecidable sentences in its language. But then, Carnap's equivalence 'theorem' could not be correct either (and its proof was, again, flawed).

The main significance of Gödel's incompleteness results lies in these implications. In addition, they were important because of certain notions and techniques introduced in proving them. For instance, Gödel provided an explicit characterization of 'primitive recursive' and 'recursive functions', notions later central for computability

<sup>15</sup> This brief summary glosses over several details, e.g., the fact that Gödel worked with  $\omega$ -consistency, not with consistency, at a crucial point (a detail amended by Rosser soon thereafter). Moreover, a proof of the second incompleteness theorem was not included in Gödel's 1931 paper; it was supposed to appear in 'Part II', which was never published. Hilbert and Bernays's *Grundgesetze der Mathematik, Vols. I-II* (1934, 1939) filled that gap in print. For a recent, technical, and detailed treatment of Gödel's theorems, see Smith (2007). For more background, cf. again Mancosu, Zach, and Badesa (2009), also George and Velleman (2002).

<sup>16</sup> More precisely again, for higher-order logic no deductive system exists that is adequate relative to standard set-theoretic semantics. (The situation changes if one allows for other kinds of semantics, e.g., those provided by Henkin models or by category theory; for the latter, cf. Awodey and Reck 2002b.)

<sup>14</sup> For more on Tarskian 'methodology of the deductive sciences', see Blok and Pigozzi (1988). For the sense in which Tarski and Carnap were at the cutting edge of logic, cf. Awodey and Reck (2002a).

theory.<sup>17</sup> His ingenious way of talking about expressions indirectly (in terms of 'Gödel numbers'), together with the technique of 'coding' proof-theoretic facts within arithmetic (the 'arithmetization of syntax'), also proved useful more generally. Tarski recognized quickly that they could be employed to establish a result he had conjectured before: the indefinability of the set of (Gödel numbers of) true sentences in arithmetic. More generally, Tarski and Carnap were among the first to be aware of the revolutionary character of Gödel's results. They also helped with their dissemination—Tarski by lecturing on them in Warsaw right after receiving Gödel's letter in January of 1931, Carnap by discussing them in his next book, *Logische Syntax der Sprache*.

## 17.5 CARNAP AND THE LOGICAL SYNTAX OF LANGUAGE

After learning about the fundamental problems with his 1920s approach, Carnap quickly regrouped and began working on a new project in logic. This project incorporated several big changes in his outlook. Some of them were prompted directly by Tarski and Gödel: he now worked self-consciously with the distinction between object-level and meta-level; his new approach was in line with the incompleteness theorems; and he used Gödel's technique of arithmetizing syntax at certain points. But there were other, more original, and quite radical changes as well. The two most important ones, for present purposes, are: First, Carnap explicitly abandoned the idea of working within just one (privileged, universal) logical system; instead, a whole range of such systems was to be explored. Second, none of these systems was seen as even potentially 'the correct' one, in any metaphysical or strong foundational sense; they were all just more or less useful. Taken together, what Carnap thus adopted was a kind of 'pluralistic pragmatism', a distinctive move.

The two changes just mentioned are closely related. In *Logische Syntax der Sprache* (1934) they were also tied to a third feature, already flagged in the title of the work: Carnap's 'syntactic' methodology. The guiding idea here was that philosophical disputes could be addressed in a productive way, and many of them resolved, by switching from the 'material mode of speech', in which they had traditionally been formulated, to the 'formal mode of speech'. The latter was not only seen as less misleading, but also as amenable to logical, and especially syntactic, analysis. This idea was not entirely without precedents. In fact, it was influenced by Hilbert's meta-mathematics, in which a central goal was to turn vague philosophical debates about the foundations of mathematics

into precise mathematical questions. What *Logische Syntax* added was to apply this idea much more widely—philosophy in general was to be done by studying the 'logical syntax of language' (in the pluralistic and pragmatist manner indicated above).

While Carnap promoted a 'syntactic' approach quite generally, his main application in *Logische Syntax* was more specific. He used it to mediate in the debates about the foundations of mathematics raging at the time, i.e., the disagreements between logicians, formalists, and intuitionists. For that purpose Carnap distinguished two languages, simply called 'Language I' and 'Language II'. Language I is a version of primitive recursive arithmetic, devised to capture the neutral core of mathematics acceptable even to intuitionists. Language II, much stronger and intended to be sufficient for all of 'classical mathematics', is a version of simple type theory superimposed on unrestricted arithmetic. Along Carnap's lines, both languages could be studied 'syntactically'. Prior philosophical arguments about them were to be put aside and, instead, their pragmatic merits weighed. To repeat, neither language was supposed to be 'the correct' one; they were just more or less useful, relative to whatever goal or goals one was pursuing.<sup>18</sup>

Besides this comparison of Languages I and II, *Logische Syntax* contains several other details worth noting in our context. For instance, during Carnap's discussion of Gödel's first incompleteness theorem in the book he simplified the theorem's proof in a now standard way (by introducing what has come to be called the 'Fixed Point Lemma'). In his attempt to capture the notion of 'analyticity' syntactically, Carnap partly anticipated Tarski's later analysis of logical truth (in semantic terms). And in his reflections on the relationship of logical and mathematical notions, Carnap touched on an issue that was to play an explicit role in Tarski's later investigations, namely: '[A] precise clarification of the logical symbols in our sense into logical symbols in the narrower sense and mathematical symbols has so far not been given by anyone' (p. 327). (We will come back to the latter two issues below, in Sections 17.6 and 17.8, respectively.)

With the position adopted in *Logische Syntax*, Carnap had moved far away from the logicism promoted by Frege and Russell earlier. Most crucial is his explicit rejection of the view that there is a 'correct logic' in which all reasoning is to be reconstructed; instead, there are various such systems, all to be evaluated pragmatically. This makes Carnap's position much more conventionalist and deflationist than Frege's and Russell's. At the same time, he continued to think of himself as a 'logician' in some sense. Besides pursuing the general goal of reconstructing scientific notions logically, what made him hold on to this label was a preoccupation he took himself to share with Frege and Russell, namely: to consider logico-mathematical languages not in isolation, but to keep their application in the sciences firmly in view.<sup>19</sup> The main goal in this connection was to

<sup>18</sup> This is the main point of Carnap's 'Principle of Tolerance', first stated at the end of § 17 of *Logische Syntax*: 'In logic there are no morals. Everyone is at liberty to build up his own logic, i.e., his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments' (Carnap 1937, p. 52).

<sup>19</sup> Concerning Carnap's logicism, cf. Bohnert (1975), the chapter by Steve Awodey in Creath and Friedman (2007), and, for a somewhat different interpretation, the chapter by Thomas Ricketts in the same volume.

<sup>17</sup> In a more encompassing survey of the development of modern logic and its connections to analytic philosophy, several other contributions to computability theory would deserve to be covered as well, especially those by Alan Turing and Alonzo Church (the notion of Turing machine, the  $\lambda$ -calculus, corresponding technical results, etc.). See again Mancosu, Zach, and Badesa (2009) for details.

clarify the role logic plays in scientific reasoning; and the specific form this now took for Carnap was to incorporate languages such as Language I and II into more encompassing frameworks, ones that contain not just 'analytic', but also 'synthetic' sentences.

So far I have only mentioned the notion of analyticity in passing. The attempt to capture this notion, and with it those of logical and mathematical truth, in 'syntactic' terms is often seen as the central legacy of *Logische Syntax* (especially after Quine's criticisms of it). Yet Carnap's turn to pluralist pragmatism is arguably more significant, especially in retrospect. Nevertheless, a few more remarks about analyticity are in order. On this issue Carnap was strongly influenced by Wittgenstein's notion of tautology. The question was, again, how to generalize it so as to cover all of mathematics. The core of Carnap's answer was to characterize logico-mathematical truths as truths based just on the formation rules of the language at issue (as 'L-truths'). But he knew that, because of Gödel's results, he could not spell this out in terms of syntactic derivability. He started to explore a variety of alternatives, in *Logische Syntax* and later, including rudimentary semantic ideas, infinitary logic, modal logic, etc. In spite of all his efforts, he never arrived at a satisfactory solution.<sup>20</sup> Still, these attempts led to some fruitful outcomes, including Carnap's openness to Tarski's work on truth and logical consequence.

## 17.6 TARSKI ON TRUTH AND LOGICAL CONSEQUENCE

We already considered Tarski's metalogical work from the 1920s briefly. This work came to further fruition, and started to be more widely influential, in the 1930s. Most central in this connection is Tarski's well-known essay on truth: 'Der Wahrheitsbegriff in den Formalisierten Sprachen' (published 1935 in German, 1933 in the original Polish). In it, the following question is addressed head on: Is it possible to define the notion of truth for a formalized language, such as the languages of arithmetic or geometry, in precise terms? For some time now, Dedekind, Peano, Hilbert, and others had used it (the notion of truth in a mathematical structure) in their axiomatic investigations implicitly, as had Fraenkel and Carnap in the 1920s. Thus it was not a matter of introducing the notion for the first time; nor was it a matter of correcting widespread mistakes in its earlier uses. Rather, the task was to reconstruct explicitly the implicit understanding that was already there, and by doing so, to set the stage for corresponding mathematical theorems (such as Tarski's theorem about the indefinability of arithmetic truth).

<sup>20</sup> For more on these failed attempts, see again the chapter by Awoodey in Creath and Friedman (2007); for the connection to Wittgenstein's notion of tautology, cf. Awoodey and Carus (2007); and for a helpful further analysis of Carnap's use of rudimentary semantic ideas in *Logische Syntax*, cf. de Rouilhan (2009).

Tarski's approach to this issue had three basic components: to correlate the non-logical symbols of the language with specific objects, properties, and relations (as their 'interpretation'); to use the notion of satisfaction, on that basis, for defining truth at the bottom level (for atomic sentences); and then to exploit the recursive structure of the formal language (the way in which its sentences are built up, step-by-step, out of atomic ones). Together this provides, basically, what is known today as 'truth-under-interpretation'. But there are a few noteworthy differences, especially the following two: The main logical framework within which Tarski still operates is a version of the simple theory of types, not first-order logic. And the now standard idea of considering various different domains over which the variables range is not used yet, as Tarski is still working with one fixed (universal) domain.<sup>21</sup> As a consequence, what we get is not quite the 'model-theoretic' notion of truth, only something close to it. (Tarski seems to have been aware of the latter in the 1930s, but didn't adopt it fully until later; cf. Section 17.8.)

In addition to the definition of truth itself, Tarski's paper contains other noteworthy contributions. He starts by considering general desiderata for any theory of truth, which leads to the formulation of his 'T-schema': *S* is true if and only if *P*, where '*P*' can be replaced by any statement of one's language and '*S*' by any name for that statement. (Standard example: 'Snow is white' is true if and only if snow is white.) This allows Tarski to note, next, that his own definition satisfies this desideratum. Two classical laws of logic also become provable: the law of non-contradiction and the law of the excluded middle. He takes that fact to speak in favour of his approach as well. Moreover, both a diagnosis and a solution for various 'semantic' paradoxes are provided, including Richard's Paradox and, especially, the Liar Paradox. The core of the diagnosis is that ordinary languages are 'semantically closed', in the sense of containing their own meta-language (which makes it possible to form paradoxical sentences). Tarski's solution is to make sure that this is not the case for formalized languages, by distinguishing the object language one studies, a meta-language for it, a meta-meta-language, etc. With such a hierarchy of languages in place, semantic notions such as truth can always be defined 'one level up', as indicated above, but never at the same level.

Yet another aspect of Tarski's treatment of truth turned out to be the most controversial, especially within philosophy. It concerns the question of whether the formal account given in 'Wahrheitsbegriff' should be seen as neutral between traditional theories of truth (correspondence theories, coherence theories, pragmatic theories, etc.) or not. Sometimes its neutrality seems to be implied by Tarski. But at other times—including in a later, more informal, and often anthologized essay, 'The Semantic Conception of Truth and the Foundations of Semantics' (1944)—he claims that what has been provided is an analysis, indeed a defence, of a traditional 'realist' conception of truth.<sup>22</sup>

<sup>21</sup> Along such lines one can simulate domain variation to some degree (by restricting variables, via conditionalizing, to relevant predicates). The additional step, or steps, from such an approach to the later model-theoretic one is still in need of further attention, it seems to me. (Both Tarski's and Carnap's writings from the 1930s to the 1950s might be worth reconsidering in this connection.)

<sup>22</sup> Already in his 1935 essay, Tarski quotes Aristotle in this connection: "To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is and of what is not that it is not, is true" (*Metaphysics* 1011b 26).

The latter caused strong reactions, starting with the conference in Paris in 1935 where Tarski presented his definition of truth for the first time to a large audience. Several people present at that occasion, including Otto Neurath and other members of the Vienna Circle, rejected the whole approach, since they objected to the apparent reintroduction of metaphysics along such lines. Others endorsed the new treatment of truth quickly, while interpreting it in a more neutral way. The latter group included, most prominently, Carnap.

Another essay by Tarski published in the mid-1930s, besides 'Wahrheitsbegriff' (1935), is widely seen as very significant philosophically too, namely: 'Über den Begriff der logischen Folgerung' (1936). While the former provides an explicit account of truth for formalized languages, what the latter adds is a parallel account of logical consequence (in the semantic sense). Here again, Tarski formulates a precise, mathematically exploitable definition of a notion implicitly understood before. Also again, his account comes close to, but is not identical with, the current model-theoretic account (formulated explicitly in the 1950s, as we will see more below). The core idea common to both is this: A sentence  $A$  is the semantic consequence of a set of sentences  $\{B_1, B_2, \dots, B_n\}$  if and only if every interpretation that makes all the  $B_i$  ( $1 \leq i \leq n$ ) come out true also makes  $A$  true. Two differences are, like before: In the 1936 essay Tarski still works with simple type theory, not first-order logic; and he doesn't vary the domain underlying his interpretations yet. Finally, the notion of logical truth, or 'logical validity', can now be defined as a limiting case of logical consequence, as follows:  $A$  is logically true if and only if it is a logical consequence of the empty set, i.e., comes out true under all interpretations. (This is what Carnap anticipated, partly and somewhat indirectly, in his 1934 book.)

Tarski's treatments of truth and logical consequence proved hugely influential. In mathematical logic, they would soon provide the foundation for the new subdiscipline of model theory (especially in the modified forms these treatments assumed in the 1950s). And it is in that subdiscipline that the results in axiomatics discussed earlier, from Dedekind and Hilbert to Fraenkel and Carnap, can be spelled out in full precision and pushed even further. In philosophy, Tarski's accounts were taken to be highly significant as well, even if their precise and full significance remained controversial, as already indicated.<sup>23</sup> Perhaps most importantly, they came to be seen as paradigmatic examples of the logical analysis of concepts, thus reinforcing Frege's, Russell's, and Wittgenstein's influence and shaping 'analytic philosophy' in a deep way. More specifically again, Tarskian semantics—as promoted and developed further by Carnap—became extremely influential in the philosophy of language and related fields.<sup>24</sup>

<sup>23</sup> Not only Tarski's treatment of truth but also that of logical consequence has elicited controversy, although the latter started later; cf. Etchemendy (1988, 1996), more recently the essays by Etchemendy and M. Gómez-Torrente in Patterson (2008). Concerning truth, cf. the essays by S. Feferman, M. David, and P. Mancosu in Patterson (2008), also general surveys in books such as Kirkham (1995).

<sup>24</sup> For its influence in the philosophy of language, see Müller's chapter in the present volume.

## 17.7 CARNAP ON SEMANTICS, MODAL LOGIC, AND INDUCTIVE LOGIC

Tarski's work on truth and logical consequence became widely available in 1935–6, through his publications and his participation in international meetings. Carnap's *Logische Syntax* had come out shortly before that, in 1934. However, the reception of both was thwarted for a while, especially in Europe, because of political events. The rise to power of the Nazis drove many Central European philosophers and scientists into exile—including our three protagonists, who all ended up in the United States. Carnap was the first to arrive, already in 1936, taking up a position at the University of Chicago, later another at UCLA. Tarski, who followed in 1939, eventually settled down at the University of California at Berkeley. Gödel made it to the US in 1940 and became a member of the Institute for Advanced Studies in Princeton.<sup>25</sup> After their relocation, all three began to publish in English, as illustrated by the (expanded) English edition of Carnap's *Logische Syntax* (1937) and by Tarski's textbook, *Introduction to Logic and the Methodology of the Deductive Sciences* (1941, translated from Polish). Partly for that reason, their ideas became most influential in the English-speaking world.

Upon his arrival in the US, Carnap continued to write on topics he had investigated before, e.g., the well-known article 'Testability and Meaning' (1936), also his contribution to the International Encyclopedia of Unified Science, the booklet *Foundations of Logic and Mathematics* (1939). However, his main attention had turned to issues in semantics by now, largely under Tarski's influence (but also building on related ideas in his own earlier writings). Carnap's focus on semantics became fully manifest in the 1940s, with publications such as *Introduction to Semantics* (1942) and *Formalization of Logic* (1943). These books were intended to establish Tarskian ideas more firmly and to make them available more widely. Carnap also integrated these ideas into his own philosophical perspective, which thus broadened beyond his previous, narrower focus on 'syntax'. As a general result, Tarskian semantics—or a combination of syntax and semantics along Tarskian and Carnapian lines (also building on Frege, Russell, Hilbert, Gödel, etc.)—became a standard part of textbooks in logic.

Carnap expanded his perspective in other respects as well, as his next book, *Meaning and Necessity* (1947), shows. The title already indicates how he intended to proceed: first, by paying systematic attention to the notion of 'meaning' (or 'intension', as opposed to 'extension'), thereby picking up on things he had learned from Frege long ago (related to Frege's notion of 'sense'); and second, by reconsidering the notions of necessity and

<sup>25</sup> Being Jewish, Tarski was most in danger. He escaped, almost accidentally, by attending a conference at Harvard in 1939. Neither Carnap nor Gödel was Jewish, but both had many Jewish friends, including some who did not manage to escape. Carnap was also politically active on the left. And by 1946, even Gödel, who was largely apolitical, was driven out by the social and political climate in Central Europe.



possibility, thus developing a novel approach to modal logic. Concerning the former, Carnap's work was parallel to and made fruitful contact with studies by Alonzo Church (on the 'logic of sense and denotation', themselves influenced by Frege). Concerning the latter, it should be noted that, while some work on modal logic had been done before, for example by C. I. Lewis at Harvard, this part of logic was still relatively marginal at the time. That started to change with Carnap's work (and soon led to major contributions by Ruth Barcan Marcus, Saul Kripke, David Lewis, and others).

Besides extending the scope of logic, Carnap's books had a strong influence on the study of natural language, thus adding to Tarskian ideas in yet another way. In his essays from the 1930s, Tarski had indicated that, while it is impossible to formalize a language such as English in its entirety (on pain of inconsistency), significant fragments of it are amenable to such treatment. Carnap's new investigations went further—they suggested ways in which one could deal with aspects of ordinary language that had proven recalcitrant to formalization so far, especially ones involving 'intensional contexts'. (His particular suggestions were soon found wanting, however, as in other cases.) Carnap's and Tarski's contributions led to 'formal semantics'. More specifically, they set the stage for possible world semantics, Montague grammar, and somewhat later, Donald Davidson's 'truth-conditional' theory of meaning. All of this had a strong impact on the nascent field of philosophy of language, but also beyond philosophy, on fields such as linguistics and computer science.<sup>26</sup> By focusing on language as a main topic for inquiry, it also contributed to, and built on, the 'linguistic turn' in analytic philosophy.<sup>27</sup>

To round off this section, let me briefly mention three further contributions in Carnap's works from the 1940s and 1950s that are still influential today: his investigations of probability and inductive logic, in *Logical Foundation of Probability* (1950b) and *The Continuum of Inductive Methods* (1952); the refinement of his deflationary views concerning metaphysics, in 'Empiricism, Semantics, and Ontology' (1950a); and the further development of his general methodology, both in *Meaning and Necessity* (1947) and *Logical Foundations of Probability* (1950). With his writings on inductive logic, Carnap contributed to yet another extension of logic, now beyond deductive logic. Its strong impact can still be felt in 'formal epistemology' (epistemic logic, Bayesian models of scientific reasoning, etc.), and more generally, in 'formal philosophy'.<sup>28</sup> The refinement of Carnap's deflationary approach to metaphysics—his distinction between 'internal' and 'external' questions and related suggestions—was, among others, a response to W. V. O. Quine's previous resurrection of ontology. Quine's less deflationary position dominated analytic philosophy over the next few decades, especially in the US; and together with the rise of modal logic, it led to a revival of metaphysics. Yet Carnap's

<sup>26</sup> For a survey of formal semantics that emphasizes Tarski's and Carnap's influence, cf. King (2006). In computer science, I have in mind topics such as the syntax and semantics of programming languages.

<sup>27</sup> On the linguistic turn, see Hacker's chapter in the present volume.

<sup>28</sup> Cf. Hendricks (2007) as well as Hendricks and Simons (2005), respectively. Here and elsewhere, Carnap's work in logic shades over into the philosophy of science, to which he made various other contributions as well; cf. the contributions by Mormann, Demopoulos, and Zabel in Creath and Friedman (2007).

reaction to the latter has found defenders again today. Finally, the articulation of Carnap's mature methodology in terms of 'explication'—which expands on his pluralistic pragmatism—is starting to attract attention again as well.<sup>29</sup>

## 17.8 FURTHER CONTRIBUTIONS BY TARSKI AND GÖDEL

While Carnap held positions in philosophy departments after his move to the US, Tarski's and Gödel's new professional homes were in mathematical or mathematics-oriented institutes. This reflects the main impact of their later works. From the mid-1940s on, Tarski became a major force in the mathematics department at Berkeley, gradually assembling around himself the most impressive and influential group of mathematical logicians in the world.<sup>30</sup> One result of their joint work, already alluded to above, was the formation and consolidation of model theory as a subdiscipline of mathematical logic in the 1950s and '60s. It was here that the notions of interpretation, truth-in-a-model, and logical consequence were all reconceived in now standard ways: by focusing on first-order logic, allowing for variations of the domain of discourse, and working in ZF set theory as the general framework.<sup>31</sup> While Tarski continued pursuing many of his older, philosophically rooted projects at Berkeley as well—from set theory and the foundations of geometry to decision procedures and general methodological questions—there was a general tendency to move logic away from philosophy and more in the direction of mathematics.<sup>32</sup> This was a self-conscious move. A main goal was to break down the barrier between logic and advanced mathematics, not the least to convince mainstream mathematicians of logic's significance. For better or worse, this contributed to a gradual drifting apart of mathematical logic and philosophy.

There were and are exceptions to this tendency, no doubt. A good example, provided by Tarski himself, is a talk he gave in 1966, entitled 'What are Logical Notions?' (Tarski 1986b). In it, he returned to a question that had been raised in connection with Frege's and Russell's works, as well as mentioned in Carnap's *Logische Syntax*, namely: Is there a principled way of distinguishing the notions of modern logic from all other notions,

<sup>29</sup> For metaphysics, cf. the chapters by Chalmers, Eklund, Hirsch, Hofweber, and Price in Chalmers et al. (2009). For Carnap's mature methodology, cf. Carus (2007) and, more critically, Reck (2012). As Carus makes clear, the later Carnap has moved far away from the positivistic stereotype often associated with his work.

<sup>30</sup> Counting colleagues, students, and regular visitors, the list includes: Addison, Chang, Craig, Feferman, Henkin, Keisler, Monk, Montague, Robinson, Scott, Vaught, and many others.

<sup>31</sup> For more on Tarski and model theory, cf. Vaught (1986), also again Feferman and Feferman (2004).

<sup>32</sup> Here I have in mind Tarski's development of algebraic approaches to logic, his exploration of connections to measure theory, topology, etc.; cf. again Feferman and Feferman (2004), earlier also two special issues of *The Journal of Symbolic Logic* on Tarski: Vol. 51 (1986) and Vol. 53 (1988).

including mathematical ones? Another way to ask this question is: What, if anything, is special about the 'logical constants' (negation, conjunction, the existential and universal quantifier, etc.)? Tarski proposed an intriguing answer (inspired by the Erlangen Program of Felix Klein): The logical constants are distinctive in being invariant under all 1-1 mappings of the universe of discourse onto itself (under all relevant automorphisms). A philosophically interesting but controversial aspect of this proposal is that it doesn't just cover the constants of first-order logic, but also, for example, those of the simple theory of types, thus leading to an inclusive view of 'logic'. The proposal's further exploration, including suggested amendments to it, has continued until today.<sup>33</sup>

Like Tarski's, Gödel's works published after his immigration to the US had their biggest impact in mathematical logic, although Gödel remained motivated by philosophical concerns throughout. Already in the 1930s, he had turned towards axiomatic set theory as a research focus, and it was in this area that he made his next major contribution. It concerned both the Axiom of Choice and the Continuum Hypothesis. The latter (a conjecture about the cardinality of the set of real numbers) had been formulated by Cantor, in the late nineteenth century, and then highlighted by Hilbert, in 1900, as one of the main open problems in mathematics. Yet nobody had been able to prove or disprove it. Gödel's approach—presented in *The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis* (1940)—was, once more, strikingly original. He showed, not that AC and CH are provable, but that they are at least consistent relative to the usual Zermelo–Fraenkel axioms. His method was to construct an 'inner model' of the latter in which both AC and CH hold (the model  $L$  of 'constructible sets'). Doing so opened up a whole new dimension for axiomatic set theory. In 1966, Gödel's result was complemented by Paul Cohen's proof that the negations of AC and CH are also relatively consistent. Together this shows that they are independent of the ZF axioms.

Such independence results raise fundamental questions. To begin with, how should the situation in set theory now be viewed? Is it akin to geometry, where Euclidean geometry (including Euclid's Fifth Postulate) and various non-Euclidean geometries (with some form of its negation) have come to be seen as equally legitimate? In that case, neither 'Cantorian' set theory (with CH) nor 'non-Cantorian' set theories (with forms of its negation) would be true or privileged in any absolute sense. Or is such a pluralistic, relativistic view about set theory to be rejected? More basically, how could a principled decision be reached in this connection? Gödel himself suggested a direction in which to go: the study of additional axioms to decide CH, especially so-called 'large cardinal axioms' (concerning the existence of large infinities). The idea was that some of them might have a special justification, thus blocking set-theoretic relativism.

A related, more basic move by Gödel was to establish close ties between axiomatic set theory, now usually framed in first-order logic, and simple type theory (by iterating the latter's types into the transfinite, parallel to the iterative conception of set). This allowed subsuming the study of type theory under advanced set theory. It thus

reinforced a general shift in logic, present in Tarski's works from the period as well: away from a Fregean or Russellian perspective rooted in higher-order logic, and towards seeing first-order logic and axiomatic set theory as the main foundational theories for mathematics. Gödel made various other contributions to mathematical logic that influenced its development too. Still concerning set theory, he played a role in formulating the 'Von Neumann–Bernays–Gödel' (NBG) axioms, as an alternative to the Zermelo–Fraenkel axioms, which allows for a systematic treatment of (proper) classes. He also contributed to the development of proof theory, especially in two ways: by showing that classical arithmetic can be embedded in intuitionistic arithmetic (thus establishing that certain views about their relationship were untenable); and by suggesting how Hilbert's 'finitist' standpoint might be modified and extended in a fruitful way (thus possibly circumventing his own supposed 'refutation' of Hilbert's proof-theoretic program).<sup>34</sup>

Yet another side of Gödel's later works concerned philosophy more directly. From the 1940s on, he published a number of overtly philosophical essays, such as 'Russell's Mathematical Logic' (1944) and 'What is Cantor's Continuum Problem?' (1947). In them (also in related lectures and unpublished notes), he endorsed a 'Platonist' or 'realist' position, to the effect that questions like the Continuum Hypothesis have objectively true answers. Such answers were to be found by a kind of 'intuitive insight', informed by rational inquiry. This went hand in hand with Gödel's study of the works of G. W. Leibniz and, from the 1950s on, those of Edmund Husserl, whose rationalist and phenomenological approaches seemed congenial to him. Gödel's Platonist remarks provoked strong and often critical reactions, while his interest in phenomenology helped to bring Husserl back to the attention of analytic philosophers.<sup>35</sup>

## 17.9 CONTINUED INTERACTIONS AND CLASHING CONVICTIONS

Carnap's, Gödel's, and Tarski's paths continued to cross after their moves to the US, both in person and in their writings. To mention just three examples of personal contacts: In 1940–1, shortly after his arrival in the US, Tarski spent a year as a research fellow at Harvard where Carnap was also visiting at the time. In 1941–2, Tarski was a year-long visitor at the Institute of Advanced Studies in Princeton, not long after Gödel had arrived

<sup>34</sup> For more on Gödel's strong influence on set theory, see Floyd and Kanamori (2006) and Kanamori (2007); for proof theory, cf. Avigad and Feferman (1995) and Tait (2006). While at Princeton, where Einstein was his colleague, Gödel even contributed to mathematical physics, by finding a surprising solution to Einstein's field equations for gravitation; see Vol. II of his collected works.

<sup>35</sup> For both Gödel's Platonism and his interest in Husserl, see, e.g., Parsons (1995), Tieszen (1998), Van Atten and Kennedy (2003), and Hauser (2006). For more on his unpublished writings, cf. Tait (2001).

<sup>33</sup> See Sher (1991), more recently also Bonnamy (2008) and the chapter by Sher in Patterson (2008).

there. And during 1952–4, Carnap spent time at the Institute in Princeton while Gödel was there. In each case, this provided the opportunity for direct interactions. One example of a crossing of paths in writing is this: When a volume on Carnap for the *Library of Living Philosophers* series was in preparation, in the mid-1950s, Gödel was invited to contribute. He spent a considerable amount of energy on preparing an essay for it, although it was not included in the end.

An interesting aspect of these later interactions is that they reveal a considerable amount about the philosophical convictions of our three protagonists. This is noteworthy especially in the case of Tarski, who generally avoided expressing philosophical views in his publications; but Gödel too had been reluctant to do so until the 1940s. What comes to the fore, moreover, is a striking divergence of philosophical outlooks. The basic contrast between Gödel's Platonist views, as expressed in his essays from the 1940s, and Carnap's deflationary position on metaphysics should be clear. Beyond that, Gödel's planned contribution to the Carnap volume—a paper entitled 'Is Mathematics Syntax of Language?'—was to contain a direct refutation of Carnap's syntax-based approach to mathematics (which Gödel never managed to formulate in a satisfactory form, thus withholding it). During Carnap's and Tarski's discussions at Harvard, in 1940–1, a different but similarly stark contrast emerges. As we know now, Tarski had nominalist convictions (partly inherited from his Polish teachers) and he expressed them forcefully on that occasion, although this did not make Carnap change his mind. Then again, Tarski's nominalism appears to have had a significant impact on two younger philosophers also present at the Harvard discussions: Quine (of that period) and Nelson Goodman, the co-authors of 'Steps Toward a Constructive Nominalism' (1947).<sup>36</sup>

Carnap's, Gödel's, and Tarski's many contributions to logic were thus grounded in, and partly guided by, radically different metaphysical views. Evidently these differences did not prevent them from taking each other seriously and interacting fruitfully. In Tarski's case there also seems to have been an odd disconnect between his nominalist leanings and the free use of set-theoretic methods in his meta-logical work. (Unlike Hilbert, he never restricted the means to be used at the meta-level.) Gödel always formulated his mathematical results in a way that was philosophically as neutral as possible so as not to restrict their reception. (His careful formulations of the incompleteness theorems are a good example.) Nevertheless, he was clearly motivated by philosophical convictions in his research, probably from early on. Carnap, finally, seems to have valued the mathematical expertise of both Gödel and Tarski so much that he was able to put aside their metaphysical views (as he had done with the Platonism of his teacher Frege). He also usually tried to mediate between opposed viewpoints by focusing on formal aspects, thus navigating around metaphysical quagmires.

<sup>36</sup> For Gödel's aborted criticism of Carnap in the 1950s, see Vol. 3 of Gödel's collected works. For Carnap's and Tarski's discussions at Harvard, including Tarski's nominalism, cf. Mancosu (2005, 2008). For Tarski's more general engagement with philosophy, cf. Woleński (1993) and Mancosu (2009).

## 17.10 CONCLUDING REMARKS

There can be no doubt about the importance of Gödel's and Tarski's contributions to the development of modern logic. They proved theorems that are among the most famous and influential in the field. They also played decisive roles in reorienting, or even creating, entire subfields of logic, such as set theory and model theory. In contrast, Carnap is seldom acknowledged as a major contributor to logic. As no specific results in mathematical logic can be connected with his name, this is understandable. Nor did any of his systematic projects in logic work out fully or result in definitive treatments, the way in which Gödel's and Tarski's did. Carnap clearly didn't have their mathematical abilities. He also wasn't as good an expositor of logic as Gödel or Tarski, both of whom were masters at it, although he still promoted logic effectively. All of this applies especially to the core areas of mathematical logic: set theory, model theory, and proof theory. But if one includes modal, intensional, and inductive logic as well, Carnap's role in the history of logic is harder to ignore. And even with respect to the core areas he played an important historical role, since he influenced Tarski and, especially, Gödel directly.

With respect to philosophy the perception tends to be the opposite. Carnap has to be covered in any respectable history of twentieth-century philosophy. His pluralistic pragmatism and his notion of explication are lasting contributions to philosophical methodology (although crude stereotypes of him as a 'positivist' still sometimes prevent their recognition). He also influenced formal semantics, metaphysics, and epistemology in profound ways. Then again, Gödel and Tarski played important roles in the development of analytic philosophy too, as we saw. In Gödel's case, this was ensured already by his incompleteness theorems, which establish something crucial about the limits of formal reasoning (perhaps also about the limits of the mind, although that is controversial).<sup>37</sup> In addition, his Platonist views provoked strong reactions, mostly of a critical kind. In the case of Tarski, three kinds of philosophical influence deserve highlighting: his reshaping of our views on logic, along meta-theoretic lines; his thorough impact on the philosophy of language; and the fact that his accounts of truth and logical consequence came to be seen as paradigms of logical analysis.

Overall, our conception of logic was transformed profoundly in the period from the 1920s to the 1950s. This includes the rise of meta-logic, with its sharp distinction between syntactic and semantic notions and techniques. There was also the separation and further exploration, not only of type theory (simple and ramified), first-order logic, propositional logic, and axiomatic set theory, but also of intuitionistic logic, intensional logic, modal logic, and inductive logic. Logic came into its own as a subfield of mathematics, with the rise of proof theory, model theory, and advanced set theory. Several classic positions in the philosophy of mathematics were reconceived and refined: logicism, formalism, and intuitionism, also Platonism, nominalism, and

<sup>37</sup> For controversies concerning the broader implications of Gödel's famous results, cf. Franzén (2005).

deflationism. Finally, logical tools found innovative and far-reaching applications in various branches of philosophy, as well as in linguistics and related fields, thereby affecting methodology greatly. Both individually and as a group, Carnap, Gödel, and Tarski played central roles in all of these developments. In fact, it is hard to even imagine what modern logic and much of analytic philosophy would look like without their contributions.<sup>38</sup>

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